Hydrogen in Electrodynamics. VII
The Pauli Principle

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Z. Naturforsch. **46a,** 869–872 (1991); received June 28, 1991

After a discussion of the one-component Schrödinger (1926) and the four-component Dirac (1928) representation of hydrogen it is shown that the six-component electrodynamic picture turns out to be considerably simpler and clearer. The computational effort is reduced to a fraction.

1. Historical Comment on the Pauli Principle

The Pauli Principle has been discovered in 1925 by Wolfgang Pauli through a comparison of various term spectra [1]. This rule influences the whole of atomistics, especially quantum physics, it underlies the periodic system of elements and thus is the governing principle of chemical sciences. Beyond that, the principle runs through our knowledge about nature as one of the great mysteries, especially since up to now we failed, despite considerable efforts, to find its foundation within the framework of the sciences concerned – or even to imagine its causal origin.

The problem is evident, and the question is so harmless that it would be accessible to any popular-science defence:

In the particle picture: Two Kepler-systems (two two-body problems) in the same state can not belong to the same concentric system unit; two Kepler systems in different states, however, may.

In the wave picture: Two Kepler-systems (two two-wave field problems) in the same state can not belong to the same concentric system unit; two Kepler systems in different states, however, may.

The question pending for half a century is: Why is that so?

The quantum people have doubtlessly added something in the course of the half of a century passed since then, though almost nothing came to light comparable to the ground-breaking discoveries of the first quarter of our century. After all, the probabilistic theory has enabled a handy translation of our quantum knownhow into technology, and it has advanced into the subatomic range that far by means of symmetry arguments that the objects of its investigation already may not be considered as separable any more. On the other hand it can not be denied that the Kopenhagen interpretation has negated immediate and manifest pieces of information of wave mechanics, as, e.g., the numerical and other primary statements of the Schrödinger or Dirac functions, respectively, only because these facts can not be correlated to the now as ever disputed notion of position probability. In particular, on the one hand, most immediate information gets lost in the probability picture, and then, on the other hand, much farther back, only the vague contours of symmetry properties are available for classification and comparison with experimental data. Thus such a function-theoretically important property as orthogonality, for instance, turns out to be of no significance in the light of the Kopenhagen interpretation. Here, for electrodynamic hydrogen, we shall establish the Pauli principle with the help of this eminent property of the eigenfunctions within just a few computational steps.

2. The Hydrogen Solution

For this purpose we again start with covariant source-free electrodynamics [2] (1)

$$\left\{\begin{matrix} \text{rot}\,\boldsymbol{E}+\frac{\mu}{c}\,\dot{\boldsymbol{H}}=0\\[4pt] \text{rot}\,\boldsymbol{H}-\frac{\varepsilon}{c}\,\dot{\boldsymbol{E}}=0\\[4pt] \text{div}\,\varepsilon\boldsymbol{E}=0\\[4pt] \text{div}\,\mu\boldsymbol{H}=0 \end{matrix}\right\}_{\text{div}\,\boldsymbol{E}=\text{div}\,\boldsymbol{H}=0} \equiv \left\{\begin{matrix} \text{rot}\,\boldsymbol{E}+\frac{\mu}{c}\,\dot{\boldsymbol{H}}=0\\[4pt] \text{rot}\,\boldsymbol{H}-\frac{\varepsilon}{c}\,\dot{\boldsymbol{E}}=0 \end{matrix}\right\}_{\substack{\text{div}\,\boldsymbol{E}=\text{div}\,\boldsymbol{H}=0\\ \boldsymbol{E}\,\perp\,\text{grad}\,\varepsilon\\ \boldsymbol{H}\,\perp\,\text{grad}\,\mu}} \tag{1}$$

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and insert the hydrogen-interfraction

$$(\varepsilon_{\text{int}}^{\text{hyd}}, \mu_{\text{int}}^{\text{hyd}}) = \left(1 - \frac{\Phi - m_0 c^2}{\hbar\omega}, 1 - \frac{\Phi + m_0 c^2}{\hbar\omega}\right) \tag{2}$$
$$= \left(1 - \frac{-e_0^2/r - m_0 c^2}{\hbar\omega}, 1 - \frac{-e_0^2/r + m_0 c^2}{\hbar\omega}\right).$$

The general solution of this system yields, according to [2] (59), (61), the hydrogen field as

$$E_{\pm}^{\text{hyd}} = \begin{pmatrix} 4CR_E^R P_l^{|m|} \cos\theta\, e^{i(m-1/2)\phi}\, e^{i(\pm\phi/2-\omega t)} \\ i\,4CR_E^R P_l^{|m|} \cos\theta\, e^{i(m-1/2)\phi}\, e^{i(\pm\phi/2-\omega t)} \\ 4CR_E^R P_l^{|m|} \sin\theta\, e^{i(m+1/2)\phi}\, e^{i(\pm\phi/2-\omega t)} \end{pmatrix}, \tag{3}$$

$$H_{\pm}^{\text{hyd}} = \begin{pmatrix} 4CR_H^R P_l^{|m|} \cos\theta\, e^{i(m-1/2)\phi}\, e^{i(\pm\phi/2-\omega t)} \\ i\,4CR_H^R P_l^{|m|} \cos\theta\, e^{i(m-1/2)\phi}\, e^{i(\pm\phi/2-\omega t)} \\ 4CR_H^R P_l^{|m|} \sin\theta\, e^{i(m+1/2)\phi}\, e^{i(\pm\phi/2-\omega t)} \end{pmatrix}. \tag{4}$$

Here one has to bear in mind especially that the exponential functions, of course, must not be contracted, since this would amount to a cancellation of the ambiguity of the azimuthal functions resulting from the solution.

In (3) and (4) the double-sign indices indicate the two possible spin states. Restricting ourselves to one of them, for example to the left-spinning alternative, and combining, we get

$$Q^{\text{hyd}} = 4CR_Q^R P_l^{|m|}\, e^{i(m+1/2)\phi}\, e^{i(\phi/2-\omega t)} \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix},$$
$$Q = E, H, \tag{5}$$

if we assign a negative sign to the third component according to the arguments between [2] (62) and (63). According to (1), the Q are centrally transverse, so that we have

$$Q^{\text{hyd}} \cdot \boldsymbol{r}^0 = Q^{\text{hyd}} \cdot \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = 0 \quad (\boldsymbol{r}^0\text{: radius vector}). \tag{6}$$

The condition (6), to begin with, is trivially fulfilled because the azimuthal functions are ambiguous in sign and therefore cancel each other. One easily sees, however, that it is fulfilled even if one omits the two azimuthal functions.

Here one should remember [4] (5)–(17). There we have proved (in a one-component calculation) that the real and imaginary parts of (5) represent two wave fields whose centers of energy together form the inherent Kepler system.

Computing with (5) the scalar product $\boldsymbol{E} \cdot \boldsymbol{H}^* = \boldsymbol{E}^* \cdot \boldsymbol{H}$, or the vector product $\boldsymbol{E} \times \boldsymbol{H}^* = \boldsymbol{E}^* \times \boldsymbol{H}$, one sees that the electric and magnetic vectors are directed parallel to each other in hydrogen, as it must be, since no Poynting radiation is allowed to occur.

Electrodynamics thus perceives the basic difference between light and (neutral) matter as a difference in the relative directions of $\boldsymbol{E}$ and $\boldsymbol{H}$: for light the electric and magnetic vectors are perpendicular, for matter they are parallel to each other.

Because of the ambiguity in sign of the two azimuthal functions, caused by their half-integer arguments

$$\overset{\pm}{e}{}^{i(m+1/2)\phi} = \begin{pmatrix} \overset{+}{e}{}^{i(m+1/2)\phi} \\ \overset{-}{e}{}^{i(m+1/2)\phi} \end{pmatrix} = \begin{pmatrix} \overset{+}{e}{}^{i(m+1/2)\phi} \\ -\overset{+}{e}{}^{i(m+1/2)\phi} \end{pmatrix} \tag{7}$$

and

$$\overset{\pm}{e}{}^{i(\phi/2-\omega t)} = \begin{pmatrix} \overset{+}{e}{}^{i(\phi/2-\omega t)} \\ \overset{-}{e}{}^{i(\phi/2-\omega t)} \end{pmatrix} = \begin{pmatrix} \overset{+}{e}{}^{i(\phi/2-\omega t)} \\ -\overset{+}{e}{}^{i(\phi/2-\omega t)} \end{pmatrix}, \tag{8}$$

the question arises whether the fields (5) annihilate each other by “self-interference” in such a way that they vanish identically. – Writing out the ambiguity in (5) we get

$$Q^{\text{hyd}} = CR_Q^R P_l^{|m|} (\overset{+}{e}{}^{i(m+1/2)\phi} - \overset{+}{e}{}^{i(m+1/2)\phi})$$
$$\cdot (\overset{+}{e}{}^{i(\phi/2-\omega t)} - \overset{+}{e}{}^{i(\phi/2-\omega t)}) \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \tag{9}$$

where we have put the factor 4 into the constant C. From (9) follows

$$Q^{\text{hyd}} = 2CR_Q^R P_l^{|m|} (\overset{+}{e}{}^{i(m+1/2)\phi}\, \overset{+}{e}{}^{i(\phi/2-\omega t)}$$
$$- \overset{+}{e}{}^{i(m+1/2)\phi}\, \overset{+}{e}{}^{i(\phi/2-\omega t)}) \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}. \tag{10}$$

The hydrogen field thus consists of the two oppositely parallel vector terms of equal magnitude

$$\overset{\pm}{Q}{}^{\text{hyd}} = \pm 2CR_Q^R P_l^{|m|}\, \overset{+}{e}{}^{i(m+1/2)\phi}\, \overset{+}{e}{}^{i(\phi/2-\omega t)} \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \tag{11}$$

which oscillate synchronously in opposite directions in their common spherically tangential plane due to their central transversality (Figure 1). – That the two vector fields (11) do not annihilate each other but have to be viewed as enduring counter-oscillating parts of the solution follows from [3] (1). According to this the

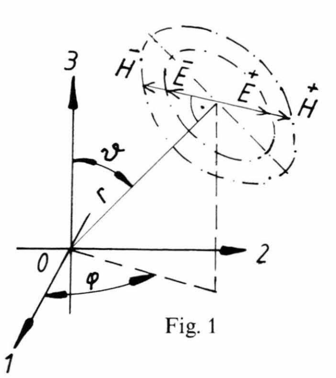

Fig. 1

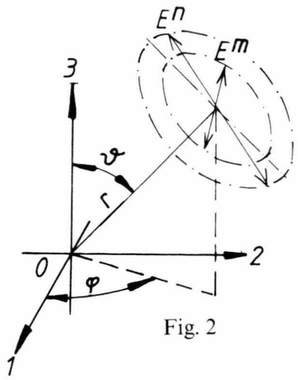

Fig. 2

hydrogen field (5) has the total energy

$$U^{\text{hyd}} = \tfrac{1}{2}(\varepsilon^{\text{hyd}} \boldsymbol{E}^{\text{hyd}} \boldsymbol{E}^{*\text{hyd}} + \mu^{\text{hyd}} \boldsymbol{H}^{\text{hyd}} \boldsymbol{H}^{*\text{hyd}}) = C^2 [(R_E^R)^2 + (R_H^R)^2] (P_l^{|m|})^2 \neq 0\,, \tag{12}$$

where we again have put the factor 4 into the constant C.

In calculating the energy (12), the two braches (11) of (5) are squared separately and only then added. This is a consequence of the clearly evident fact that a period of the spin function does not extend over an arc of 2π but over the double arc of 4π.

Although the field components thus vanish according to (10), the field is there since its energy is there, and radiation can not take place because of the parallel alignment of $\boldsymbol{E}^{\text{hyd}}$ and $\boldsymbol{H}^{\text{hyd}}$. Particle fields vanishing in this way (like a dipole) do not show their electromagnetic quality from a macroscopic point of view and manifest their existence only through the inertia of their energy (12). And that, because the center of the hydrogen field is governed by Newtonian (or some modified) mechanics. This fact we demonstrated for the Schrödinger-hydrogen in [4] (5)–(17).

Borrowing from the concept of a dipole we shall call oppositely oscillating half-spin vector fields "divector fields" from now on.

3. Establishing the Pauli Principle

By turning a compass needle $\boldsymbol{N}$ in a stationary magnetic field $\boldsymbol{H}$, it alternatively gets into unstable and stable states of equilibrium every 180°. We therefore have a rotational potential

$$D = |\boldsymbol{N}|\,|\boldsymbol{H}| \cos\alpha = \boldsymbol{N} \cdot \boldsymbol{H} \quad (\alpha\text{: angle of rotation}) \tag{13}$$

and a torque of magnitude

$$|M| = \frac{\mathrm{d}D}{\mathrm{d}\alpha} = -|\boldsymbol{N}|\,|\boldsymbol{H}| \sin\alpha = |\boldsymbol{N} \times \boldsymbol{H}|\,. \tag{14}$$

For the interaction between oppositely rotating divector fields the situation is somewhat different. From Fig. 2 one can see that a doubling in the succession of the interactions results from the doublesided oscillations of the divector, if two centrally transverse divectors, as, e.g., $\boldsymbol{E}^n$ and $\boldsymbol{E}^m$, rotate relatively to each other. To this succession doubling there corresponds a squaring of the trigonometric function in (13), so that we now have a rotational potential of

$$D = |\boldsymbol{E}^n|\,|\boldsymbol{E}^m| \cos^2\alpha = \boldsymbol{E}^n \cdot \boldsymbol{E}^m \cos\alpha \tag{15}$$

and a torque of magnitude

$$|M| = \frac{\mathrm{d}D}{\mathrm{d}\alpha} = -2\,|\boldsymbol{E}^n|\,|\boldsymbol{E}^m| \sin\alpha\cos\alpha = 2\,|\boldsymbol{E}^n \times \boldsymbol{E}^m| \cos\alpha\,. \tag{16}$$

Accordingly, $\boldsymbol{E}^n$ and $\boldsymbol{E}^m$ get into the stable state for $\cos\alpha = 0$, that is, for $\alpha = \pi/2, 3\pi/2, \ldots$, each time therefore for orthogonal positions.

Concentrated hydrogen solutions thus can remain stable within each other if and only if they are orthogonal to each other. – In order that the Pauli principle be established we therefore have to show that the electromagnetic field strength eigenfunctions of different states are orthogonal to each other.

For this purpose we start from the set (5) of the hydrogen eigenfunctions and imagine that one and only one eigenfunction belongs to each energy level (the later adjustment to spin and momentum degeneracy can easily be performed in closed form).

The differential equations of second order of the system (1), or the set (5), respectively, are of the well-known form

$$\left(\Delta-\frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}-\frac{1}{\mu}\operatorname{grad}\mu\times\operatorname{rot}\right)\boldsymbol{E}=0, \tag{17}$$

$$\left(\Delta-\frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}-\frac{1}{\varepsilon}\operatorname{grad}\varepsilon\times\operatorname{rot}\right)\boldsymbol{H}=0, \tag{18}$$

or rewritten according to (1)

$$\left(\Delta-\frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{E}+\frac{1}{c}\operatorname{grad}\mu\times\frac{\partial}{\partial t}\boldsymbol{H}=0, \tag{19}$$

$$\left(\Delta-\frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{H}-\frac{1}{c}\operatorname{grad}\varepsilon\times\frac{\partial}{\partial t}\boldsymbol{E}=0. \tag{20}$$

From this, in view of (5), with

$$\dot{\boldsymbol{E}}=-i\omega\boldsymbol{E},\quad \ddot{\boldsymbol{E}}=-\omega^2\boldsymbol{E}\quad\text{and}$$

$$\dot{\boldsymbol{H}}=-i\omega\boldsymbol{H},\quad \ddot{\boldsymbol{H}}=-\omega^2\boldsymbol{H}$$

we get

$$\left(\Delta+\omega^2\frac{\varepsilon\mu}{c^2}\right)\boldsymbol{E}-i\frac{\omega}{c}\operatorname{grad}\mu\times\boldsymbol{H}=0, \tag{21}$$

$$\left(\Delta+\omega^2\frac{\varepsilon\mu}{c^2}\right)\boldsymbol{H}+i\frac{\omega}{c}\operatorname{grad}\varepsilon\times\boldsymbol{E}=0, \tag{22}$$

For simplicity we now put

$$\frac{\omega^2\varepsilon\mu}{c^2}=a. \tag{23}$$

Writing the system (21)–(22) first for the state *n*, then its complex conjugate for the state *m*, and multiplying suitably by complex or, respectively, complex conjugate field strengths we get

$$\boldsymbol{E}^{*m}(\Delta+a_n)\,\boldsymbol{E}^n-i\frac{\omega_n}{c}\operatorname{grad}\mu_n\times\boldsymbol{H}^n\,\boldsymbol{E}^{*m}=0, \tag{24}$$

$$\boldsymbol{H}^{*m}(\Delta+a_n)\,\boldsymbol{H}^n+i\frac{\omega_n}{c}\operatorname{grad}\varepsilon_n\times\boldsymbol{E}^n\,\boldsymbol{H}^{*m}=0, \tag{25}$$

$$\boldsymbol{E}^{n}(\Delta+a_m)\,\boldsymbol{E}^{*m}+i\frac{\omega_m}{c}\operatorname{grad}\mu_m\times\boldsymbol{H}^{*m}\,\boldsymbol{E}^{n}=0, \tag{26}$$

$$\boldsymbol{H}^{n}(\Delta+a_m)\,\boldsymbol{H}^{*m}-i\frac{\omega_m}{c}\operatorname{grad}\varepsilon_m\times\boldsymbol{E}^{*m}\,\boldsymbol{H}^{n}=0. \tag{27}$$

In order to eliminate the cross products we add, taking into account (2), firstly (24) and (27), then (25) and (26), and then we subtract the two sums. This yields

$$\begin{aligned}(\boldsymbol{E}^n\,\Delta\boldsymbol{E}^{*m}-\boldsymbol{E}^{*m}\,\Delta\boldsymbol{E}^n)-(a_n-a_m)\,\boldsymbol{E}^n\,\boldsymbol{E}^{*m}&\\=(\boldsymbol{H}^n\,\Delta\boldsymbol{H}^{*m}-\boldsymbol{H}^{*m}\,\Delta\boldsymbol{H}^n)-(a_n-a_m)\,\boldsymbol{H}^n\,\boldsymbol{H}^{*m}&.\end{aligned} \tag{28}$$

Because of $\boldsymbol{E}\neq\boldsymbol{H}$ there follows

$$(\boldsymbol{Q}^n\,\Delta\boldsymbol{Q}^{*m}-\boldsymbol{Q}^{*m}\,\Delta\boldsymbol{Q}^n)=(a_n-a_m)\,\boldsymbol{Q}^n\,\boldsymbol{Q}^{*m},\quad(\boldsymbol{Q}=\boldsymbol{E},\boldsymbol{H}). \tag{29}$$

Integrating now over the whole of configuration space, the left-hand side vanishes according to Green's theorem. On the right-hand side we insert from (23) and (2) to get

$$\begin{aligned}(\omega_n-\omega_m)\int_\infty Q^n\,Q^{*m}\,\mathrm{d}V&\\=(\omega_n-\omega_m)\int_\infty Q^n\,Q^{*m}\,\frac{2\Phi}{\hbar(\omega_1+\omega_2)}\,\mathrm{d}V&.\end{aligned} \tag{30}$$

Because of the monotonicity of the hydrogen potential function Φ (see (21)) this relation can only be fulfilled in case that both sides vanish. Thus we have

$$(\omega_n-\omega_m)\int_\infty Q^n\,Q^{*m}\,\mathrm{d}V=0. \tag{31}$$

Since the vanishing can not occur through the radial summation it has to result through the azimuthal or polar integration, or through both. One can easily convince oneself with the help of the $t-\phi$-symmetry of the spin factors that the cancellation of the interaction comes about through the azimuthal integration. The vanishing shows itself as a consequence of the spins. The integral (31), therefore, not only vanishes as a space integral, but already as line integral along any parallel circle of latitude. Any twist of the two fields out of their concentric-coaxial basic position thus makes the right-hand side of (31) different from zero.

Equation (31) states that two concentric hydrogen fields in different states – are orthogonal to each other and, therefore (vide (16)), mutually are in a stable state. And that two hydrogen fields in the same state – prevent their concentration by means of an interaction in the amount of their total energy.

The Pauli principle herewith appears to be established.

[1] W. Pauli, Z. Phys. **31,** 765 (1925).
[2] H. Sallhofer, Z. Naturforsch. **45a,** 1361 (1990).
[3] H. Sallhofer, Z. Naturforsch. **34a,** 1145 (1979).
[4] H. Sallhofer, Z. Naturforsch. **43a,** 1039 (1988).